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Inequalities for Ultraspherical Polynomials and the Gamma Function*

LEE LORCH

Mathematics Department, York University, Downsview, Ontario M3J 1P3, Canada

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Let $M_n^{(\lambda)} = (n+\lambda)^{1-\lambda} \max_{0 \leqslant \theta \leqslant \pi} (\sin\theta)^\lambda \, |P_n^{(\lambda)}(\cos\theta)|$, where $P_n^{(\lambda)}(x)$ is the ultraspherical polynomial of degree n and parameter λ. It is shown that $M_n^{(\lambda)} < 2^{1-\lambda}/\Gamma(\lambda)$, for $0 < \lambda < 1$ and $n = 0, 1, 2, \ldots$. When $\lambda = 0$ and when $\lambda = 1$, this inequality becomes an equality. It refines inequality (7.33.5) of G. Szegö's "Orthogonal Polynomials" (4th edition, 1975, p. 171), wherein the factor $(n+\lambda)^{1-\lambda}$ is replaced by $n^{1-\lambda}$. The method of proof requires sharpening some inequalities for the ratio $\Gamma(n+\lambda)/\Gamma(n+1)$, $n = 0, 1, 2, \ldots$.

For the ultraspherical (Gegenbauer) polynomial of degree n and parameter λ, $0 < \lambda < 1$, $P_n^{(\lambda)}(x)$, a standard inequality [7, Theorem 7.33.2, p. 171], will be strengthened, as will some related inequalities for the gamma function. For $P_n^{(\lambda)}(x)$ it will be proved, i.a., that

$$(n+\lambda)^{1-\lambda} (\sin\theta)^\lambda \, |P_n^{(\lambda)}(\cos\theta)| < 2^{1-\lambda}/\Gamma(\lambda), \qquad 0 \leqslant \theta \leqslant \pi, \tag{1}$$

$n = 0, 1, 2, \ldots$, $0 < \lambda < 1$. This makes more precise the customary inequality [7, (7.33.5), p. 171] in which is found the factor $n^{1-\lambda}$ instead of $(n+\lambda)^{1-\lambda}$ in (1).

Indeed, somewhat more than (1) will be shown; cf. (3), (5) and (6) below.

For the important special case of Legendre polynomials, where $\lambda = \frac{1}{2}$ and the right-hand member of (1) becomes $(2/\pi)^{1/2}$, (1) has been established by V. A. Antonov and K. V. Holševnikov [1]. Their proof uses complex variable methods. Later [6] it was shown that their (strengthened) inequality for Legendre polynomials can be demonstrated also by using, in somewhat sharpened form, the real variable method by which S. N. Bernstein obtained the standard form of that inequality; his result is stated and his method presented in [7, Theorem 7.3.3, p. 165].

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It is this method which will be employed here, again suitably sharpened, to prove (1) and the more detailed results (5) and (6) below.

1. On the Gamma Function

To effectuate these proofs some information on the gamma function needs to be made more precise. In particular, we shall need

$$\beta_k \equiv \frac{(2k+\lambda)^{1-\lambda}\,\Gamma(k+\lambda)}{\Gamma(k+1)} \uparrow 2^{1-\lambda}, \qquad \text{as} \quad k \to \infty, \tag{2}$$

k a non-negative integer, $0 < \lambda < 1$.

That the limit in (2) is $2^{1-\lambda}$ is obvious from a fundamental limit relation for the gamma function [5, p. 15], taking into account that

$$\frac{(2k+\lambda)^{1-\lambda}}{(2k)^{1-\lambda}} \to 1, \qquad \text{as} \quad k \to \infty.$$

To complete the proof of (2), therefore, it is necessary only to show that $\beta_{k+1}/\beta_k > 1$, $k = 0, 1, 2, \ldots$.

From the functional equation $\Gamma(x+1) = x\Gamma(x)$ it follows that

$$\frac{\beta_{k+1}}{\beta_k} = \left(\frac{2k+\lambda+2}{2k+\lambda}\right)^{1-\lambda} \frac{k+\lambda}{k+1} \equiv \rho(k).$$

Clearly, $\rho(k) \to 1$ as $k \to \infty$. Once it is demonstrated that $\rho(k)$ is a decreasing function of k for fixed λ, $0 < \lambda < 1$, then (2) is established. Letting $k \geqslant 0$ be a continuous variable, we have

$$\begin{aligned}\rho'(k)[(k+1)^2\,(2k+\lambda)^{2-\lambda}\,(2k+\lambda+2)^{\lambda}]\\ = \lambda(1-\lambda)(\lambda-2) < 0, \qquad 0 < \lambda < 1.\end{aligned}$$

This completes the proof of (2).

2. An Inequality for Ultraspherical Polynomials

The inequality (1), and its refinements (5) and (6) below, will be derived as consequences of (2). To make the transition, define

$$M_n^{(\lambda)} \equiv (n+\lambda)^{1-\lambda} \max_{0 \leqslant \theta \leqslant \pi} (\sin\theta)^{\lambda}\,|P_n^{(\lambda)}(\cos\theta)|, \qquad 0 < \lambda < 1. \tag{3}$$

In this notation, what needs to be proved is

$$M_n^{(\lambda)} < 2^{1-\lambda}/\Gamma(\lambda), \qquad 0 < \lambda < 1, \quad n = 0, 1, 2, \ldots. \tag{4}$$

This is a consequence of the pair of still more informative inequalities, namely,

$$M_{2k-1}^{(\lambda)} < M_{2k}^{(\lambda)}, \qquad k = 1, 2, \ldots, 0 < \lambda < 1, \tag{5}$$

and

$$M_{2k}^{(\lambda)} \uparrow 2^{1-\lambda}/\Gamma(\lambda), \qquad \text{as} \quad k\uparrow, 0 < \lambda < 1, k = 0, 1, 2, \ldots. \tag{6}$$

To establish (5), we note from [7, Theorem 7.33.2, and the comment following immediately after the statement of the theorem, p. 171] that

$$\begin{aligned} M_{2k-1}^{(\lambda)} &< \frac{2k(2k-1+\lambda)^{1-\lambda}\,\alpha_k}{[4k^2-4(1-\lambda)k+1-\lambda]^{1/2}} \\ &= \left(\frac{2k-1+\lambda}{2k+\lambda}\right)^{1-\lambda} \frac{2k}{[4k^2-4(1-\lambda)\,k+1-\lambda]^{1/2}} (2k+\lambda)^{1-\lambda}\,\alpha_k \\ &= \left(\frac{2k-1+\lambda}{2k+\lambda}\right)^{1-\lambda} \frac{2k}{[4k^2-4(1-\lambda)k+1-\lambda]^{1/2}} M_{2k}^{(\lambda)}, \end{aligned}$$

$k = 1, 2, \ldots.$

Here [7, (4.9.21), p. 93]

$$\alpha_k = \binom{k+\lambda-1}{k} = \frac{\Gamma(k+\lambda)}{\Gamma(\lambda)\,\Gamma(k+1)},$$

and [7, Theorem 7.33.2, p. 171]

$$M_{2k}^{(\lambda)} = (2k+\lambda)^{1-\lambda}\,\alpha_k. \tag{7}$$

To prove (5) it suffices to show that the square of the coefficient of $M_{2k}^{(\lambda)}$ in the inequality connecting $M_{2k-1}^{(\lambda)}$ and $M_{2k}^{(\lambda)}$ is less than one.

With $m = 2k$, the square of this coefficient becomes

$$\varphi(m) = \left(\frac{m-1+\lambda}{m+\lambda}\right)^{2-2\lambda} \frac{m^2}{m^2-2(1-\lambda)m+1-\lambda}.$$

Obviously, $\varphi(m) \to 1$ as $m \to \infty$, so that (5) will follow from showing that $\varphi(m)$ increases with m, for fixed λ, $0 < \lambda < 1$. Now,

$$\begin{aligned} &(m+\lambda)^{3-2\lambda}\,[m^2-2(1-\lambda)m+1-\lambda]^2\,\varphi'(m) \\ &\quad = 2\lambda(1-\lambda)(m)(m-1+\lambda)^{1-2\lambda}\,[(2-\lambda)m-1+\lambda], \end{aligned}$$

so that

$$\varphi'(m) > 0 \qquad \text{for} \quad m > (1-\lambda)/(2-\lambda),\ 0 < \lambda < 1,$$

a condition obtaining here, since $m = 2, 4, 6, \ldots$.

Thus, $\varphi(m) \uparrow 1$ as $m = 2k$ increases through the positive even integers, and (5) is proved.

It remains to demonstrate (6). It is here that the property of the gamma function (2) will be used.

Reverting to [7, Theorem 7.33.2, especially (7.33.4), and comment, p. 171], we observe that

$$M_{2k}^{(\lambda)} = [\Gamma(\lambda)]^{-1} \beta_k.$$

Thus, (6) is merely a rewording of (2), already established, and so has been proved.

Together with (5), this completes the proof of (1), and supplies the indicated extra information about $M_n^{(\lambda)}$, $0 < \lambda < 1$.

3. Remarks

1. For $\lambda = 0$ and for $\lambda = 1$, the inequality (4) becomes an equality.

The inequality (1) cannot be improved by replacing $(n+\lambda)^{1-\lambda}$ by $(n+\lambda+\varepsilon)^{1-\lambda}$ for any constant $\varepsilon > 0$, for any $\lambda, 0 < \lambda < 1$. This follows from the asymptotics for the gamma function applied to (7) for $M_{2k}^{(\lambda)}$, $k = 0, 1, 2, \ldots$. For $\lambda = \frac{1}{2}$, this was pointed out in [1].

2. The standard gamma function asymptotics make it clear that in (2) the factor $2k + \lambda$ cannot be replaced by $2k + \lambda + \varepsilon$ for any constant $\varepsilon > 0$. Numerical calculations suggest that the inequality implicit in (2), rewritten to give the upper bound in (8), is rather precise. Indeed, we have

$$\frac{1}{(k+\lambda)^{1-\lambda}} < \frac{\Gamma(k+\lambda)}{\Gamma(k+1)} < \frac{1}{(k+\frac{1}{2}\lambda)^{1-\lambda}}, \qquad 0 < \lambda < 1, \tag{8}$$

$k = 0, 1, 2, \ldots$, where both bounds are closer to the gamma function ratio than those provided by W. Gautschi's inequalities [2, (7)].

To verify the lower bound, let

$$\gamma_k = (k+\lambda)^{1-\lambda}\, \Gamma(k+\lambda)/\Gamma(k+1), \qquad r(k) = \gamma_{k+1}/\gamma_k,$$

and consider k to be a continuous variable. It can be shown that $r'(k) > 0$, $0 < \lambda < 1$, $k \geqslant 0$, so that $r(k)\uparrow$, $k \geqslant 0$. Hence $\gamma_{k+1} < \gamma_k$, $k = 0, 1, 2, \ldots$, since $r(k) \to 1$ as $k \to \infty$. But also $\gamma_k \to 1$ as $k \to \infty$, whence $\gamma_k \downarrow 1$, $k = 0, 1, 2, \ldots$, thereby establishing the lower bound in (8).

This lower bound, and indeed (8) as a whole, has been improved by D. Kershaw [4], according to the information that the referee of the present paper has kindly transmitted to me. Kershaw proves

$$\frac{1}{[k+(\lambda+\frac{1}{4})^{1/2}-\frac{1}{2}]^{1-\lambda}} < \frac{\Gamma(k+\lambda)}{\Gamma(k+1)} < \frac{1}{(k+\frac{1}{2}\lambda)^{1-\lambda}}, \qquad 0<\lambda<1, \qquad (9)$$

where $k>0$ is a continuous variable, not merely integer-valued as in (8). The referee advised me further that [4] contains inequalities even sharper than (9).

In the special case in which $\lambda=\frac{1}{2}$, the inequalities (8) are already known, established first by D. K. Kazarinoff [3] and then by G. N. Watson [8], via proofs valid for k continuous. Watson's lower bound is bigger than the one in (9) and a fortiori than the one in (8) for $\lambda=\frac{1}{2}$.

If $\lambda>1$, then $r'(k)<0$, so that $\gamma_k \uparrow 1$ and we have the upper bound in

$$(k+\tfrac{1}{2}\lambda)^{\lambda-1} < \frac{\Gamma(k+\lambda)}{\Gamma(k+1)} < (k+\lambda)^{\lambda-1} \qquad (10)$$

for all $\lambda>1$. The lower bound is valid only for $1<\lambda<2$, since $\rho'(k)>0$, $1<\lambda<2$ and $\rho'(k)<0$, $\lambda>2$. Thus, the two-sided inequality (10) holds only for $1<\lambda<2$ and $k=0, 1, 2,...$.

For $\lambda>2$,

$$\frac{\Gamma(k+\lambda)}{\Gamma(k+1)} < (k+\tfrac{1}{2}\lambda)^{\lambda-1}, \qquad k=0, 1, 2,.... \qquad (11)$$

3. As the Legendre case ($\lambda=\frac{1}{2}$) shows, the sequence $\{M_n^{(\lambda)}\}$, $n=0, 1, 2,...$, unlike the sequence $\{M_{2k}^{(\lambda)}\}$, $k=0, 1, 2,...$, is not increasing, at least not from the beginning [6, Remark 1]. Judging from the numerical work reported in [6] for the case $\lambda=\frac{1}{2}$, it would seem reasonable to guess that $\{M_{2k-1}^{(\lambda)}\}$, $k=1, 2,...$, is an increasing sequence.

(*Note added in proof, September* 19, 1983.) Inequality (1) is implicit also in the more general inequality (23) in L. Durand's paper (Nicholson-type integrals for products of Gegenbauer functions and related topics, *in* "Theory and Application of Special Functions" (R. Askey, Ed.), Academic Press, New York, 1975, pp. 353–374, esp. p. 362). To make the transition from the appropriate special case of Durand's inequality (23) to (1) it suffices to apply to his definition (21) the upper bound in inequality (8) of this paper or, more generally (i.e., in the case where n in (1) is not restricted to integer values), to use the upper bound in Kershaw's inequality (9).

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